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The Dirac equation in de Sitter space

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Abstract. It is shown that Dirac's equation for a particle of spin one-half in de Sitter space can be derived by the use of a simple group-theoretic argument. The significance of this is discussed.

1. Introduction

It was shown by Dirac (1935) that one could construct a plausible wave equation for particles of spin one-half in those particular curved spaces called de Sitter spaces. The argument used was a somewhat heuristic one, and it is the aim of this paper to show that the same equation may be derived by a simple group-theoretic argument, in which the crucial step is merely to take the difference of the values assigned to a Casimir element of a certain Lie algebra in two simple representations.

2. de Sitter space

A discussion of the physical significance of de Sitter spaces can be found in the first two references (Dirac 1935, Gürsey 1964). We content ourselves here with observing that they represent two of the five simplest spatially homogeneous, isotropic and temporally invariant models for space-time in which Einstein's field equations obtain.

Both models may be realized as pseudo-spheres in pseudo-Euclidean spaces:

$$x_1{}^2+x_2{}^2+x_3{}^2-x_4{}^2+x_5{}^2 = R^2$$

in a space with diagonal metric $(-1,\ -1,\ -1,\ +1,\ -1)$, and

$$x_1{}^2+x_2{}^2+x_3{}^2-x_4{}^2-x_5{}^2 = R^2$$

in a space with diagonal metric $(-1,\ -1,\ -1,\ +1,\ +1)$.

Both are homogeneous spaces in the mathematical sense, admitting the transitive symmetry groups SO(4, 1) and SO(3, 2) respectively. These groups are often called the de Sitter groups for this reason. In either case the subgroup fixing a point can be identified with the Lorentz group SO(3, 1).

For convenience of discussion Dirac makes some of the coordinates complex so that he may deal with the two types of de Sitter space simultaneously (Dirac 1935). This is done in such a way that they may both be conceived as a four-sphere, S:

$$x_1{}^2+x_2{}^2+x_3{}^2+x_4{}^2+x_5{}^2 = R^2$$

in a five-dimensional Euclidean space. The relevant groups are now SO(5) acting transitively on S, and SO(4) fixing a point. We shall do likewise, the argument being in no way seriously distorted as a result of this.

3. The de Sitter group and its Lie algebra

Since we are interested in the group-theoretic aspects of quantum mechanics in these spaces, it is clearly incumbent upon us to investigate the representations of SO(5) and those differential operators invariant under its actions. We are particularly interested in those representations of SO(5) which can be used to describe particles localizable in S. For these purposes it is convenient to use the concept of localizability due to Mackey and Wightman (Mackey 1963, Wightman 1962). If this is done then the 'imprimitivity theorem' of Mackey informs us that these are precisely the representations induced from representations of SO(4).

The simplest way in which to construct operators invariant under these divers group actions is to exploit the properties of the Lie algebra SO(5). This has a structure very similar to that of its three-dimensional counterpart, the Lie algebra of the rotation group. It is generated by the ten linearly independent elements of the set $\{J_{ab}: a, b = 1, 2, ..., 5\}$, satisfying

$$J_{ab}+J_{ba} = 0$$

and the commutation relations

$$[J_{ab}, J_{cd}] = i(\delta_{ad}J_{bc}-\delta_{ac}J_{bd}+\delta_{bc}J_{ad}-\delta_{bd}J_{ac}).$$

The centre of this Lie algebra is two-dimensional. Using the summation convention we may write down a basis of two Casimir elements:

$$J^2 = J_{ab}J_{ab}$$

and

$$W^2 = W_aW_a$$

where

$$W_a = \mathscr{E}_{abcde}J_{bc}J_{de}.$$

($\mathscr{E}_{abcde}$ is the usual alternating symbol.)

These correspond to the two elements which take values m^2 and $m^2s(s+1)$ in representations of the Poincaré group, the Lie algebra of which is a contraction of the above.

4. Invariant differential operators

Any representation of SO(5) naturally gives rise to a representation of the Lie algebra, and the central elements are represented by invariant operators on the representation space.

For example, the simplest interesting case is when we induce a representation of SO(5) from the identity representation of SO(4). Here the representation is defined on the space of square-integrable complex-valued functions on S. (These can be interpreted as the wave functions for a spin-zero particle in S if we require a physical picture.) The generators J_{ab} of the Lie algebra are mapped to the angular momentum operators, $M_{ab} = i\{x_a(\partial/\partial x_b)-x_b(\partial/\partial x_a)\}$, and J^2 becomes $M^2 = M_{ab}M_{ab}$, which is essentially the Laplace–Beltrami operator for S. Having this invariant differential operator at our disposal we can write down a Klein–Gordon equation for spinless particles in S:

$$M^2\Psi = \lambda\Psi.$$

In general we can obtain a second-order invariant differential operator for any representation by taking the image of J^2. It is therefore an easy task to find second-order wave equations invariant under a given representation of the group SO(5). However, there are good reasons for requiring differential operators occurring in the wave equations of quantum mechanics to be of the first order. With this in mind we shall call any first-order differential operator commuting with the action of a group in a representation U, a *Dirac operator* for the representation. The wave equations introduced by Dirac for Minkowski space, and later for de Sitter space (Dirac 1935), can be put into forms in which they are eigenvalue equations for differential operators. These operators are then Dirac operators in the sense of the above definition.

We shall now show that Dirac operators can be found for quite a wide class of representations of the de Sitter group by a very simple procedure.

Suppose that we induce a unitary representation of SO(5) from the restriction to SO(4) of a finite-dimensional representation Σ of SO(5). The permanence relation for group representations tells us that the resulting representation of SO(5) is equivalent to the representation $U^1\times\Sigma$, where U^1 is the representation of SO(5) induced from the trivial representation of SO(4). (The permanence relation says that if Σ' is the restriction of a representation, Σ, of G, to a subgroup H, and R is a representation of H then $U^{R\times\Sigma'} \simeq U^R\times\Sigma$, U^R being the representation of G induced from R.) Functions which transform according to such a representation of a group, $(U^1\times\Sigma)$, are usually called manifestly covariant, and an operator which commutes with the action of such a representation is called a manifestly covariant operator. The operators constructed by Dirac which we mentioned earlier are manifestly covariant operators.

The infinitesimal generators of such a product representation are easily computed. The infinitesimal generator of a one parameter subgroup of a Lie group in a representation is just the derivative of the action at the identity element, and, for a product action, differentiating by parts we see that we get the sum of two terms. In the case in hand, $(U^1 \times \Sigma')$, we discover that the operator representing the infinitesimal generator of rotations in the 'a–b' plane is the sum $M_{ab} \times 1 + 1 \times \sigma_{ab}$, where M_{ab} is the representative of the infinitesimal generator associated with U^1, which we introduced before, and σ_{ab} is the corresponding element in the representation Σ. σ_{ab} is itself independent of position, and contains no differential operator terms. We shall frequently abbreviate the above expression for the operator representing J_{ab} to $M_{ab} + \sigma_{ab}$, as is the custom.

The first Casimir invariant, J^2, in this representation is therefore

$$(M_{ab} + \sigma_{ab})(M_{ab} + \sigma_{ab}) = M_{ab}M_{ab} + [\sigma_{ab}, M_{ab}]_+ + \sigma_{ab}\sigma_{ab}.$$

However, the two terms σ_{ab} and M_{ab}, standing as they do for the tensor products $1 \times \sigma_{ab}$ and $M_{ab} \times 1$, commute so that their anticommutator can be rewritten in any of the forms

$$[\sigma_{ab}, M_{ab}]_+ = 2M_{ab} \times \sigma_{ab} = 2M_{ab}\sigma_{ab} = 2\sigma_{ab}M_{ab}$$

if we so wish. Our Casimir invariant can therefore be written as

$$M^2 + 2\sigma_{ab}M_{ab} + \sigma_{ab}\sigma_{ab}.$$

But now we notice that the operator $M^2 \times 1$ (or in its abbreviated form just M^2) commutes with the product representation $U^1 \times \Sigma$, owing to its decomposed form. It therefore happens that the difference, $2\sigma_{ab}M_{ab} + \sigma_{ab}\sigma_{ab}$, also commutes with the group action. Better still it is a first-order differential operator, and so is a Dirac operator. It is also manifestly covariant. It enables us to write down a first-order differential equation which is manifestly covariant under the action of the group:

$$(\sigma_{ab}M_{ab} + \tfrac{1}{2}\sigma_{ab}\sigma_{ab})\Psi = \mu\Psi$$

and this can be taken as the wave equation describing a particle of a given non-zero spin.

In many cases an even simpler equation can be found, for it often happens that $\sigma_{ab}\sigma_{ab}$ is a multiple of the identity operator in 'spin' space, and so can be absorbed into the numerical constant μ. In particular this happens when we are interested in the case of a particle of spin one-half. This will have its transformation properties described by the action induced from the 'spin' representation of SO(4). We can, however, represent this spin action of SO(4) as a restriction of that of SO(5), and so use the above theory. If we evaluate the σ_{ab} in this case it turns out that

$$4i\sigma_{ab} = [\alpha_a, \alpha_b]$$

where

$$2\delta_{ab} = [\alpha_a, \alpha_b]_+.$$

That is the spin terms are commutators of representative elements of the Clifford algebra of five-dimensional Euclidean space. A simple calculation shows that $\sigma_{ab}\sigma_{ab}$ is just 5 times the identity element, so that we have at our disposal the Dirac operator $\sigma_{ab}M_{ab}$. If we multiply this by i, then we get the related invariant operator $\frac{1}{4}[\alpha_a, \alpha_b]M_{ab}$, which is precisely that used by Dirac (1935).

It is of interest to note that acting on the space of solutions of the Dirac equation

$$\tfrac{1}{4}[\alpha_a, \alpha_b]M_{ab}\Psi = Rm\Psi$$

both Casimir elements take definite values. This is because both can be factorized into expressions linear in the Dirac operator. Dirac (1935) shows that

$$\tfrac{1}{2}M^2 = (\sigma_{ab}M_{ab})(\sigma_{ab}M_{ab}+3).$$

Therefore the image of $\frac{1}{2}J^2$ in this representation is

$$\begin{aligned}\tfrac{1}{2}(M^2+2\sigma_{ab}M_{ab}+\sigma_{ab}\sigma_{ab}) &= (\sigma_{ab}M_{ab})(\sigma_{ab}M_{ab}+4)+\tfrac{5}{2}\\ &= (\sigma_{ab}M_{ab}+2)^2-\tfrac{3}{2}.\end{aligned}$$

We likewise find the image of W^2 to be a simple multiple of

$$(\sigma_{ab}M_{ab}+\tfrac{3}{2})(\sigma_{ab}M_{ab}+\tfrac{5}{2})$$

and our assertion about the Casimir operators has been justified.

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